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100 to less than 10 as the rod approaches saturation ( $H = 150$  gauss) and to decrease thereafter asymptotically to zero.

If  $p$  is the force per square centimeter of section of the rod and  $E$  Young's modulus,

$$p = E(\Delta l/l) \quad (1)$$

regarding the magnetic stress as traction.

Using the expression for the potential of a disc, the field  $F$  in a narrow crevasse normal to  $F$ , between molecular layers of magnetic surface density of magnetization  $kH$

$$F = 4\pi kH + H$$

where  $k$  is the susceptibility of the metal.

Hence the force per square centimeter should be  $p' = FkH$ , or

$$p' = H^2(4\pi k^2 + k) \quad (2)$$

Equating  $p = p'$  in equations (1) and (2)

$$\frac{\Delta l}{l} = \frac{4\pi k^2 + k}{E} H^2 \quad (3)$$

If the data in figure 3 are taken above 800 gauss, supposing that these are far enough removed from the initial complications, the estimate would be ( $E = 2 \times 10^{12}$ ),  $k = 1.6$ .

An order of mean susceptibility of 1.6 (which seems not an unreasonable assumption) would thus account for the observed contractions. Naturally as  $k$  is essentially variable with  $H$  a better statement of the case might be given by postulating such a relation.

<sup>1</sup> *London, Phil. Mag.*, 37, 1894, (131).

<sup>2</sup> *Carnegie Inst., Washington, Pub.*, No. 149.

<sup>3</sup> These PROCEEDINGS, 5, 1919, (39).

<sup>4</sup> These PROCEEDINGS, 4, 1918, (328).

## GROUPS INVOLVING ONLY TWO OPERATORS WHICH ARE SQUARES

BY A. G. MILLER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

Communicated by E. H. Moore, May 2, 1919

The abelian group of order  $2^m$  and of type  $(1, 1, 1, \dots)$  is completely characterized by the fact that all of its operators have a common square. When we impose the condition that the operators of a finite group  $G$  have two and only two distinct squares then  $G$  must belong to one of three infinite systems of groups whose characteristic properties we proceed to determine.

All the operators of  $G$  besides the identity must be of order 2 or of order 4 and  $G$  must involve operators of each of these two orders. Hence the order of  $G$  is of the form  $2^m$ . When  $G$  is abelian it is of type  $(2, 1, 1, \dots)$  and it will therefore be assumed in what follows that  $G$  is non-abelian. The octic group and the quaternion group constitute well known illustrations of such a group and have the smallest possible order.

When the operators of order 2 contained in  $G$  together with the identity constitute a subgroup this subgroup is the central of  $G$  and hence  $G$  belongs to the system of groups called Hamiltonian by R. Dedekind.<sup>1</sup> In this case it is known that  $G$  is the direct product of the quaternion group and an abelian group of order  $2^\alpha$  and of type  $(1, 1, 1, \dots)$ . Hence it will be assumed in what follows that  $G$  involves non-commutative operators of order 2.

Every operator of order 4 contained in  $G$  is transformed either into itself or into its inverse by every operator of  $G$  and an operator of order 2 contained in  $G$  has at most two conjugates under the group.<sup>2</sup> Let  $H_1, H_2$  represent subgroups composed respectively of all the operators of  $G$  which are commutative with two non-commutative operators of order 2  $s_1, s_2$ . The cross-cut  $K_1$  of  $H_1$  and  $H_2$  is of index 4 under  $G$  and includes the central of  $G$ . A set of independent generators of  $G$  can be so selected as to include  $s_1, s_2$  and operators from  $K_1$ .

Exactly one-half of the operators of  $G$  which are not also in  $K_1$ , are of order 2 since the quotient group  $G/K_1$  is abelian. If  $K_1$  involves non-commutative operators of order 2 two such operators  $s_3, s_4$  may be selected from  $K_1$  in exactly the same way as  $s_1$  and  $s_2$  were selected from  $G$ . The remaining operators of a set of independent generators including  $s_1, s_2, s_3, s_4$  may be selected from an invariant subgroup of index 4 under  $K_1$  and of index 16 under  $G$  all of whose operators are commutative with each of the four operators already chosen.

As  $G$  is supposed to be of finite order we arrive by this process at a subgroup  $K_m$  in which all the operators of order 2 are commutative. Hence  $K_m$  belongs to one of the following three well known categories of groups. Abelian and of type  $(1, 1, 1, \dots)$ , abelian and of type  $(2, 1, 1, \dots)$ , or Hamiltonian of order  $2^a$ . The commutator subgroup of  $G$  is of order 2.

In each case,  $G$  may be constructed by starting with  $K_m$ , forming the direct product of  $K_m$  and an operator  $t_1$  of order 2, and then extending this direct product by means of an operator  $t_2$  of order 2 which is commutative with each of the operators of  $K_m$  and transforms  $t_1$  into itself multiplied by the commutator of order 2 contained in  $G$ . When  $K_m$  is Hamiltonian or abelian and of type  $(2, 1, 1, \dots)$  this commutator is determined by  $K_m$ . In the other possible case it may be selected arbitrarily from the operators of order 2 found in  $K_m$ .

When  $m > 1$ , we use the group  $K_{m-1}$  just constructed in exactly the same way as  $K_m$  was used in the preceding paragraph. The commutator of order 2 is completely determined for each of the categories by  $K_{m-1}$ ,  $m > 1$ . When  $m > 2$

we proceed in the same manner with  $K_{m-2}$ , etc. It may be noted that in each of the groups belonging to one of the three categories thus constructed more than one-half of the operators are of order 2, in those belonging to the second category the number of operators of order 2 is one less than one-half of the order of  $G$ , while in those belonging to the third category the number of operators of order 2 is obtained by subtracting from one-half the order of  $G$  one plus one-fourth the order of  $K_m$ .

Some of these results constitute a proof of the following theorem: *If only two of the operators of a group  $G$  are the squares of operators contained in  $G$  then the non-invariant operators of  $G$  have only two conjugates, each cyclic subgroup of order 4 is invariant, and  $G$  belongs to one of three categories of groups of order 2 which can be separately generated by a set of operators such that each of these operators is commutative with each of the others except at most one of them.*

When  $m$  is sufficiently large there is one and only one group belonging to each of these three categories and having a give number  $\gamma$  of pairs of non-commutative operators of order 2 in its set of independent generators when this set is obtained in the manner described above. The smallest values of  $m$  for these categories are  $2\gamma + 1$ ,  $2\gamma + 2$ , and  $2\gamma + 3$  respectively. When  $m$  has a larger value  $G$  must be the direct product of an abelian group of type  $(1, 1, 1, \dots)$  and of the minimal group having  $\gamma$  such pairs of generators and contained in the category to which  $G$  belongs.

By means of these facts it is very easy to determine the number of the groups of a given order  $2^m$  which belong to each of these three categories. This number is the largest integer which does not exceed  $\frac{m-1}{2}$ ,  $\frac{m-2}{2}$ , and  $\frac{m-3}{2}$  for the three categories respectively. In particular, the number of the distinct groups of order 128 belonging to each of these categories is 3, 2, 2 respectively, it being assumed that each of the groups in question contains at least two non-commutative operators of order 2.

In each one of these groups every two non-commutative operators of order 2 generate the octic group and every two non-commutative operators of order 4 generate the quaternion group. Moreover, every non-abelian subgroup is invariant. In two of the categories the central is composed of operators of order 2 in addition to the identity, while the central of the remaining category is of type  $(2, 1, 1, \dots)$ . Every one of these groups is generated by its operators of order 2. From the standpoint of definition and structure these categories rank among the simplest known infinite systems of non-abelian groups

<sup>1</sup> Dedekind, R., *Math. Ann.*, Leipzig, **48**, 1897, (548-561).

<sup>2</sup> Miller, G. A., *Trans. Amer. Math. Soc.*, New York, **8**, 1907, (1-13).